(Only 10 (best) counts)

Thu, Nov. 20 $\boxed{\text{Exam 3}}$

8:00 – 9:15

Arrive at least 5 min early!

Stepan Center

Format:

11 multiple choice questions

3 free response questions

Practice exams on Canvas

The exam will cover: everything from Polar Coordinates to Fundamental Thm of line integrals

Lectures 23 – 32

See also Review 6

Calculators: NOT allowed

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2+y^2=r^2$$

$$r = \rho\sin\varphi$$

$$x = \rho\sin\varphi\cos\theta$$
$$y = \rho\sin\varphi\sin\theta$$
$$z = \rho\cos\varphi$$
$$\rho^2 = x^2+y^2+z^2$$

Cylindrical (r,θ,z)	Rectangular (x,y,z)	Spherical (ρ,θ,φ)
r	$\sqrt{x^2+y^2}$	$\rho\sin\varphi$
$(4, \pi/4, 4\sqrt{3})$	$(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3})$	$(8, \pi/4, \pi/6)$
$r\sin\theta = z+2$	$y = z+2$	$\rho\sin\varphi\sin\theta = \rho\cos\varphi + 2$
$r^2+z^2 = 4$	$x^2+y^2+z^2=4$	$\rho = 2$
$r = 2$	$x^2+y^2=4$	$\rho\sin\varphi = 2$

$$\iiint_E F(x,y,z)\,dV = \iiint_S F(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

True or false?

1) $\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$? **T**

2) $\int_C \vec{F} \cdot d\vec{r} = \vec{F}(\vec{r}(b)) - \vec{F}(\vec{r}(a))$? **F**

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

3) $\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$? **T**

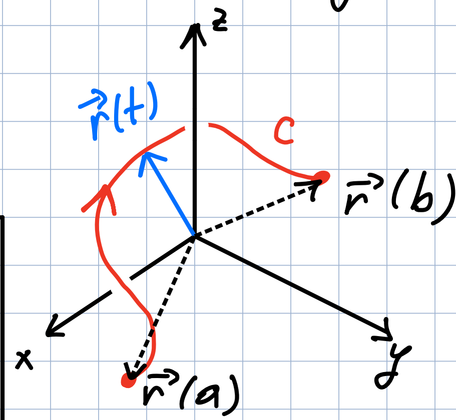
4) $\int_C f(x,y,z)\,dx + f(x,y,z)\,dy + f(x,y,z)\,dz = \int f(x,y,z)\,ds$? **F**

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))\,x'(t)\,dt$$

$$\Rightarrow \int_C f\,dx + f\,dy + f\,dz = \int_a^b f(x(t),y(t),z(t))\,(x'(t) + y'(t) + z'(t))\,dt$$

$\vec{F} = \langle P, Q, R \rangle$ - vector field

$C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$



$f(x,y,z)$ - function

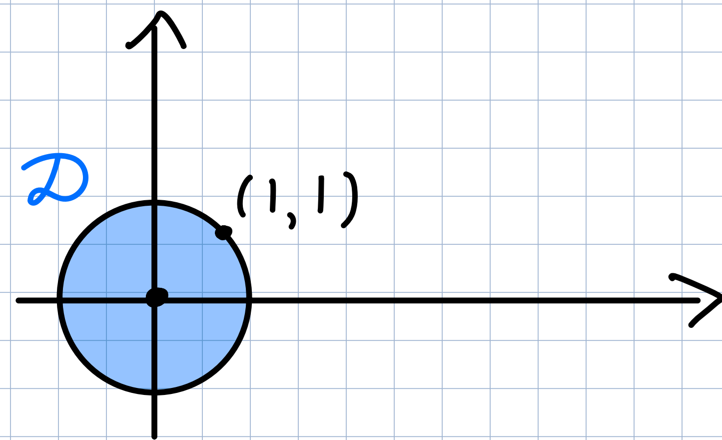
5) $\int_C \underset{\text{scalar}}{f(x,y,z)} \cdot \underset{\text{vector}}{d\vec{r}} = \int_C f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dz$? **F**

DOT PRODUCT is NOT defined!

6) $\int_C \nabla\vec{F} \cdot d\vec{r} = \vec{F}(\vec{r}(b)) - \vec{F}(\vec{r}(a))$? **F** Gradient of a vector field does NOT exist.

Rewrite in polar coordinates:

$$\iint_D \frac{x}{y} + 4x^2 \, dA$$



$$D = \{ (x,y) : \; x^2+y^2 \le 2 \}$$

$$= \left\{ (r,\theta) : \begin{array}{l} 0 \le r \le \sqrt{2} \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\iint_D \frac{x}{y} + 4x^2 \, dA = \int_0^{2\pi} \int_0^{\sqrt{2}} (\operatorname{ctg}\theta + 4r^2\cos^2\theta)\, r \, dr \, d\theta$$

$$\begin{cases} u = 3y - 2x \\ v = 3x - 2y \end{cases}$$

a) Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$

$$\begin{cases} 2u + 3v = 5x \\ 3u + 2v = 5y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{5}(2u+3v) \\ y = \frac{1}{5}(3u+2v) \end{cases}$$

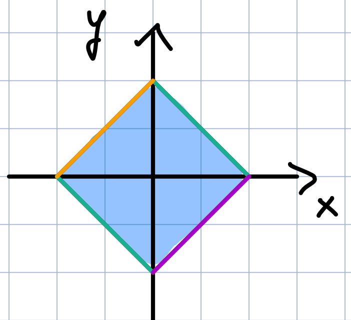
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{2}{5}\cdot\frac{2}{5} - \frac{3}{5}\cdot\frac{3}{5} = -\frac{1}{5}$$

b) $$\iint_R x + y \, dA = \iint_S \qquad du\,dv$$

$$= \iint_S (u+v)\left|-\frac{1}{5}\right| du\,dv = \iint_S \frac{u+v}{5}\,du\,dv$$

$x = r\cos\theta$
$y = r\sin\theta$

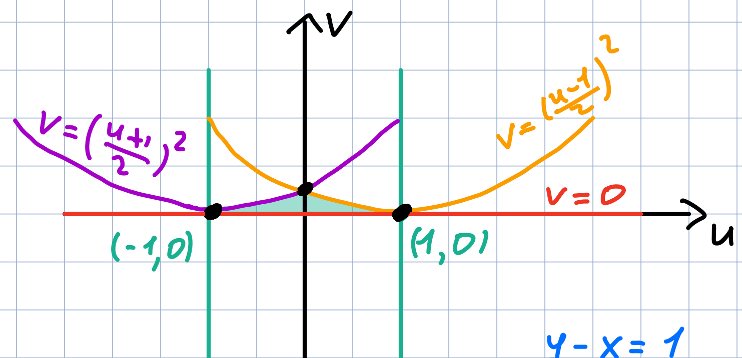
$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$



$$\begin{cases} u = x + y \\ v = x^2 \end{cases}$$

$v = x^2 \Rightarrow v \geq 0$

$x - y = 1 \Rightarrow y = x - 1$, $x \in (0,1)$



$$\begin{cases} u = 2x - 1 \\ v = x^2 \end{cases}$$

$$v = \left(\frac{u+1}{2}\right)^2$$

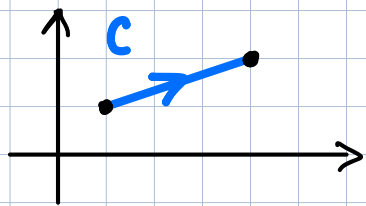
$y - x = 1$ $\quad y = x + 1$, $x \in (0,1)$

$$\begin{cases} u = 2x + 1 \\ v = x^2 \end{cases} \qquad v = \left(\frac{u-1}{2}\right)^2$$

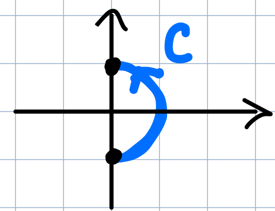
$\vec{F} = \langle \sin x,\ 2x + y \rangle$ $\qquad$ $f(x,y) = e^{xy} - y^2$

$\int_C \vec{F} \cdot d\vec{r}$ $\qquad$ $\int_C f(x,y)\, ds$ $\qquad$ $\int_C f(x,y)\, dy$

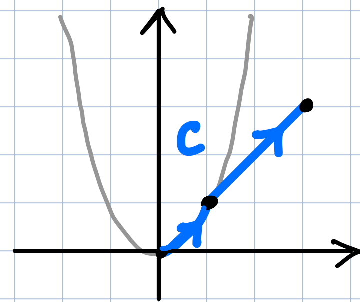
a)



b)



c)



Step 1: Parametrize C:

a) $\vec{r}(t) = \vec{r}_0 (1-t) + \vec{r}_1 t$

$\vec{r}(t) = (1-t)\langle 1,1\rangle + t\langle 4,2\rangle$

$= \langle (1-t) + 4t,\ (1-t) + 2t \rangle$

$= \langle 1+3t,\ 1+t \rangle$

$\Rightarrow\ x = 1+3t,\quad y = 1+t,\quad t \in [0,1]$

b) $\begin{cases} x = \cos t \\ y = \sin t \\ -\frac{\pi}{2} \le t \le \frac{\pi}{2} \end{cases}$

c) $C = C_1 \cup C_2$

$C_1: \begin{cases} x = t \\ y = t^2 \end{cases}$ $\quad t \in [0,1]$

$C_2: \begin{cases} x = t \\ y = t \end{cases}$ $\quad t \in [1,3]$

Step 2:

$$\int_C \vec{F} \cdot dr = \int_C P\,dx + Q\,dy = \int_a^b \Big(P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t) \Big)\, dt$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\,y'(t)\,dt$$

Step 3: a) $x'(t) = 3$, $y'(t) = 1$ b) $x'(t) = -\sin t$, $y'(t) = \cos t$ c) C_1: $x'(t) = 1$, $y'(t) = 2t$; C_2: $x'(t) = 1$, $y'(t) = 1$

Step 4: (a) $\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(3\sin(1+3t) + \left(2(1+3t) + (1+t)\right)\right)dt$

$$\int_C f(x,y)\,ds = \int_0^1 \left(e^{(1+3t)(1+t)} - (1+t)^2\right)\sqrt{9+1}\,dt$$

$$\int_C f(x,y)\,dy = \int_0^1 1\left(e^{(1+3t)(1+t)} - (1+t)^2\right)dt$$

(b) $\int_C \vec{F}\cdot d\vec{r} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-\sin(\cos t)\sin t + (2\cos t + \sin t)\cos t\right)dt$

$$\int_C f(x,y)\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(e^{\sin t\cos t} - (\cos t)^2\right)\underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2}}_{=1}\,dt$$

$$\int_C F(x,y)\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(e^{\sin t \cos t} - (\cos t)^2\right) \cos t \, dt$$

(c) $$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 (\sin t + (2t + t^2) 2t)\, dt$$

$$+ \int_1^3 (\sin t + (2t + t))\, dt$$

$$\int_C F(x,y)\,ds = \int_{C_1} F(x,y)\,ds + \int_{C_2} F(x,y)\,ds$$

$$= \int_0^1 \left(e^{t \cdot t^2} - (t^2)^2\right) \sqrt{1^2 + (2t)^2}\, dt + \int_1^3 \left(e^{t \cdot t} - t^2\right) \sqrt{1^2 + 1^2}\, dt$$

$$\int_C F(x,y)\,dy = \int_{C_1} F(x,y)\,dy + \int_{C_2} F(x,y)\,dy$$

$$= \int_0^1 \left(e^{t^3} - t^4\right) 2t\, dt + \int_1^3 e^{t^2} - t^2\, dt$$

Find the potential function of the vector field if it exists.

a) $\vec{F} = \langle 2x + 3, y - 1 \rangle$

b) $\vec{F} = \langle e^x - 2y, -2x \rangle$

c) $\vec{F} = \langle \sin x + e^y, \cos y \rangle$

(a) $P_y = 0$, $Q_x = 0$ $\Rightarrow$ F is conservative

$$\begin{cases} f_x = 2x + 3 \\ f_y = y - 1 \end{cases} \Rightarrow \quad f(x,y) = x^2 + 3x + g(y)$$

$$f_y(x,y) = 0 + 0 + g'(y) = y - 1$$

$$\Rightarrow g'(y) = y - 1 \Rightarrow g(y) = \tfrac{1}{2}y^2 - y + c$$

$$f(x,y) = x^2 + 3x + \tfrac{1}{2}y^2 - y + c$$

(b) $P_y = -2 = Q_x$ $\Rightarrow$ F is conservative

$$\begin{cases} f_x = e^x - 2y \\ f_y = -2x \end{cases} \Rightarrow \quad f(x,y) = e^x - 2xy + g(y)$$

$$f_y = -2x + g'(y) = -2x \Rightarrow g(y) = c$$

$$f(x,y) = e^x - 2xy + c$$

(c) $P_y = e^y \neq Q_x = 0$ $\Rightarrow$ There is NO potential function